Ball&Plate Model

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1 Introduction

The goal of this report is to model the physical behavior of the Ball&Plate system used in our lab and represented in Figure 1 by a mathematical formulation.

A similar work, for a ball and plate system characterized by a different mechanics, is reported in [1, 2].

Section 2 explains the physical system showing both its structure and the forces taken into account. Section 3 introduces the mathematical formulas which are used to model the dynamics of the system. Section 4 compares the simulated behavior in Simulink [3] and the dynamics of the concrete system.

Figure 1: Ball&Plate picture.

2 Physical model

In order to present the mathematical model of the ball and plate system, Figure 2 shows the xz view of the coordinate system, introducing the needed parameters to define. The yz view is identical to the xz view, considering $\hat{y}$,

l_x	l_y	D_x	D_y	H_x	H_y	h	b	r
76	62	66.5	53	47	49	16	40	20

Table 1: Plate measurements (mm).

y, $\dot{y}$, $\ddot{y}$, θ_y, ω_y, $\dot{\omega}_y$ in place of $\hat{x}$, x, $\dot{x}$, $\ddot{x}$, θ_x, ω_x and $\dot{\omega}_x$, respectively. Hence, the yz view is omitted.

The sizes of the ball and plate system (in millimeters) are reported in Table 1.

Although the real system presents the vertical axes H split in two parts, as far as we are concerned the proposed model is still valid. More precisely, the rotation joint is located along the H segment, which means that it does not coincide with the plate center which in turn is not a fixed point anymore, causing a greater θ for the same $alpha$.

A servomotor is in charge of imposing the angle α which, through the joint h, moves the plate of an angle θ. The forces, which acts on the ball, are the friction and weight forces.

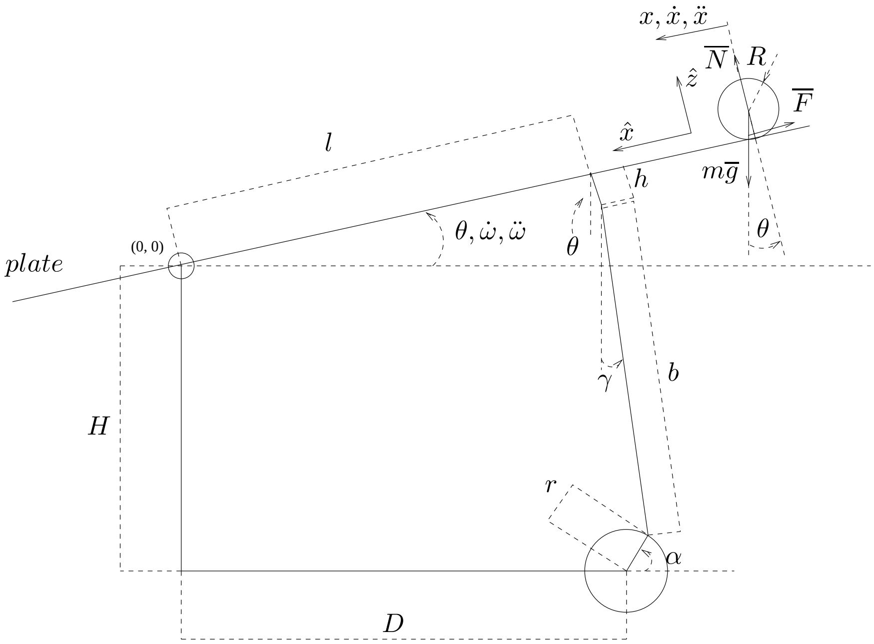

Figure 2: Physical model.

3 Mathematical model

3.1 Motion laws

The complete set of equations is continuous, non-linear and couples the two modes of motion. After a discretization and linearization process about the operating point, which is the equilibrium configuration ($x = y = \theta_x = \theta_y = 0$), the equations of motion for the ball, along the x and z axis, are the following:

$$
\begin{aligned}
\ddot{x}[k] &= \frac{5}{7}\ g\ sin(\theta_x[k]) - \left(r_b + \frac{5}{7}\ h\right)\dot{\omega}_x[k];\\
m\ddot{z}[k] &= N - mg\ cos(\theta_x[k]) = 0;\\
\dot{x}[k] &= \ddot{x}[k]\Delta t + \dot{x}[k-1];\\
x[k] &= \dot{x}[k]\Delta t + x[k-1];\\
\omega[k] &= \frac{\theta[k] - \theta[k-1]}{\Delta t};\\
\dot{\omega}[k] &= \frac{\omega[k] - \omega[k-1]}{\Delta t}.
\end{aligned}
$$

Where Δt is the updating period (the time elapsed since the previous check event until the actual). The first two equations are computed according to the Newton's laws and the last four by the classical kinematic formulas, assuming a motion uniformly accelerated during each Δt.

The final kinematic formula to model the ball motion is shown in Equation 1.

$$
\begin{aligned}
x[k] &= \ddot{x}[k]\Delta t^2 + \dot{x}[k-1]\Delta t + x[k-1] =\\
&= \left[\frac{5}{7}g\sin(\theta_x[k]) - \left(r_b + \frac{5}{7}h\right)\dot{\omega}_x[k]\right]\Delta t^2 + \dot{x}[k-1]\Delta t + x[k-1].
\end{aligned}
\tag{1}
$$

3.2 Angles

The relation between α and θ is given by the closed chain geometry (considering negligible the elastic coefficient of the plate structure).

$$
\begin{cases}
\hat{x}: \quad l\ cos(\theta) + h\ sin(\theta) = D + r\ cos(\alpha) + b\ sin(\gamma)\\
\hat{y}: \quad H + l\ sin(\theta) = r\ sin(\alpha) + b\ cos(\gamma) + h\ cos(\theta)
\end{cases}
\tag{2}
$$

The angle γ can be removed and, exploiting the trigonometric axiom $sin(\beta)^2 + cos(\beta)^2 = 1$, the problem can be rewritten as:

$$
[H + l\ sin(\theta) - r\ sin(\alpha) - h\ cos(\theta)]^2 + [l\ cos(\theta) + h\ sin(\theta) - D - r\ cos(\alpha)]^2 = b^2.
\tag{3}
$$

The relation $\theta = f(\alpha)$ which binds the servomotor angle and the plate inclination can be better formulated solving Equation 3.

$$
\begin{aligned}
1: \quad & \{[H + l\ sin(\theta)] - [r\ sin(\alpha) + h\ cos(\theta)]\}^2 + \{[l\ cos(\theta) + h\ sin(\theta)] - [D + r\ cos(\alpha)]\}^2 = b^2; \\
2: \quad & [H + l\ sin(\theta)]^2 + [r\ sin(\alpha) + h\ cos(\theta)]^2 - 2[H + l\ sin(\theta)][r\ sin(\alpha) + h\ cos(\theta)] + \\
& + [l\ cos(\theta) + h\ sin(\theta)]^2 + [D + r\ cos(\alpha)]^2 - 2[l\ cos(\theta) + h\ sin(\theta)][D + r\ cos(\alpha)] = b^2; \\
3: \quad & H^2 + l^2 sin(\theta)^2 + 2Hlsin(\theta) + r^2 sin(\alpha)^2 + h^2 cos(\theta)^2 + 2rhsin(\alpha)cos(\theta) + \\
& - 2Hrsin(\alpha) - 2lrsin(\theta)sin(\alpha) - 2Hhcos(\theta) - 2lhsin(\theta)cos(\theta) + \\
& + l^2 cos(\theta)^2 + h^2 sin(\theta)^2 + 2lhcos(\theta)sin(\theta) + D^2 + r^2 cos(\alpha)^2 + 2Drcos(\alpha) + \\
& - 2lDcos(\theta) - 2lrcos(\theta)cos(\alpha) - 2hDsin(\theta) - 2hrsin(\theta)cos(\alpha) = b^2; \\
4: \quad & b^2 = H^2 + l^2 + r^2 + h^2 + D^2 + 2Drcos(\alpha) - 2Hrsin(\alpha) + \\
& + 2cos(\theta)[hrsin(\alpha) - Hh - lD - lrcos(\alpha)] + 2sin(\theta)[Hl - lrsin(\alpha) - hD - rhcos(\alpha)].
\end{aligned}
\tag{4}
$$

In order to make the model easier, it is linearized considering small swings of θ which means that $sin(\theta) = \theta$ and $cos(\theta) = 1$. From this perspective, the result of Equation 4 can be rewritten as reported in Equation 5.

$$
\begin{aligned}
& b^2 = H^2 + l^2 + r^2 + h^2 + D^2 + 2Drcos(\alpha) - 2Hrsin(\alpha) + \\
& + 2[hrsin(\alpha) - Hh - lD - lrcos(\alpha)] + 2\theta[Hl - lrsin(\alpha) - hD - rhcos(\alpha)].
\end{aligned}
\tag{5}
$$

Expressing the formula in function of θ will lead to Equation 6.

$$
\theta = \frac{H^2 + l^2 + r^2 + h^2 + D^2 + 2Drcos(\alpha) - 2Hrsin(\alpha) - b^2 + 2[hrsin(\alpha) - Hh - lD - lrcos(\alpha)]}{2[lrsin(\alpha) + hD + rhcos(\alpha) - Hl]}.
\tag{6}
$$

Although the plate is in a rest position ($\theta = 0°$) when the servomotor is ordered to hold 0°, the actual α is not 0° according to how the ball and plate models have been designed.

α_{x0} and α_{y0} are computed, imposing $\theta = 0°$ as shown in Equation 7.

$$
\begin{cases}
\hat{x}: \quad [H - r\ sin(\alpha_{x0}) - h]^2 + [l - D - r\ cos(\alpha_{x0})]^2 = b^2 \\
\hat{y}: \quad [H - r\ sin(\alpha_{y0}) - h]^2 + [l - D - r\ cos(\alpha_{y0})]^2 = b^2
\end{cases}
\tag{7}
$$

For the sake of simplicity, the calculus refers only to x axes and it is reporter in Equation 8

$$
\begin{cases}
A\sin(\alpha) + B\cos(\alpha) + C = 0 \\
A = 2rh - 2Hr \\
B = 2Dr - 2lr \\
C = H^2 + r^2 + h^2 + l^2 + D^2 - b^2 - 2Hh - 2lD
\end{cases}
\tag{8}
$$

In order to solve the trigonometric equation imposing $t = \tan(\frac{x}{2})$ lets us transform the problem into a second order equation, as shown in Equation 9.

$$\begin{cases} t = \tan(\dfrac{x}{2}) \\ (C - B)\ t^2 + 2A\ t + B + C = 0 \end{cases} \tag{9}$$

According to the model parameters defined in Table 1, the equilibrium angles are $\alpha_{x0} \simeq -38.814°$ (-0.68 rad) and $\alpha_{y0} \simeq -30°$ (-0.52 rad).

For this reason, α needs to be consider as the sum between the rest angle and the desired angle: $\alpha = \alpha_0 + \alpha'$.

Considering small swings for $alpha$ and the addition formulas[1], trigonometric operations in Equation 6 are rewritten as:

$$\begin{aligned} sin(\alpha) =& \alpha' cos(\alpha_0) + sin(\alpha_0); \\ cos(\alpha) =& cos(\alpha_0) - \alpha' sin(\alpha_0). \end{aligned} \tag{10}$$

The plot of the function $\theta = f(\alpha)$ is depicted in Figure 3 (with $\alpha \in [0, \pi]$), where in $\alpha = 2.25$ a vertical asymptote occurs.

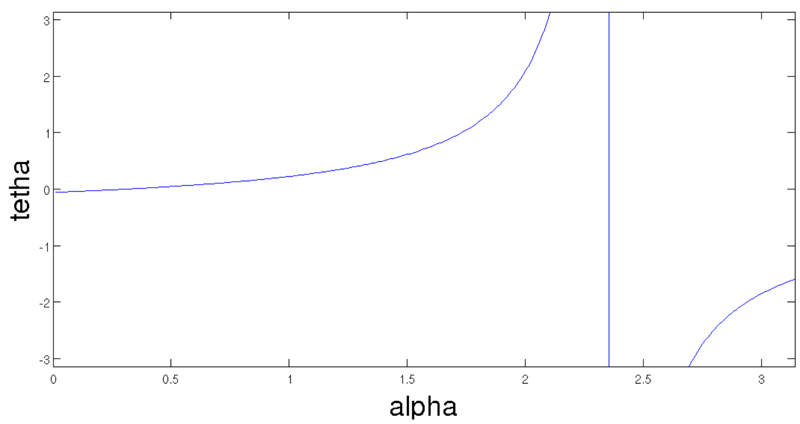

Figure 3: $\theta = f(\alpha)$.

4 Performance evaluation

This section aims at proving that the introduced mathematical model represents the ball and plate behavior by comparing the effectiveness of the same

[1]Addition formulas:

- $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$;
- $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$.

control algorithm on them. More precisely, the same controller has been tested on both the mathematical model and the concrete system and their dynamics are compared (distance with respect to the plate center).

The control algorithm is a PID with parameters: $P = 2.85$, $I = 0.7125$, $D = 2.85$ and $0.05s$ as sample time (Δt).

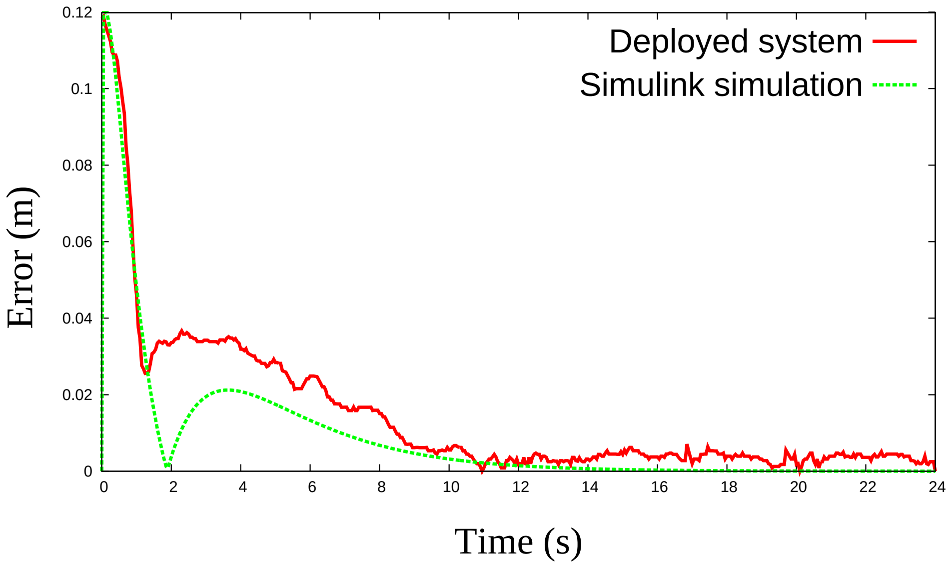

Figure 4: Performance evaluation.

As reported in Figure 4, the behavior of the simulated model is almost identical to the trend obtained by the one deployed on a concrete system.

At the first instant, the ball is placed on an edge of the plate and then it is forced close to center by the controller in less than $8s$. The error within the interval [1, 8] is due to the non-linearity of the servomotors that leads to slower of faster dynamics.

In conclusion, the similar performance between the simulated and concrete system lets us validate the mathematical and physical models of the ball and plate system taken into account in this report.

References

[1] S. Awtar and K. Craig, "Mechatronic design of a ball on plate balancing system," in *7th Mechatronics Forum International Conference, September 6-8 2000, Atlanta, Georgia*, 2000.

[2] "Awtar's website," http://www-personal.umich.edu/ awtar/MS/bop.htm.

[3] "Simulink," http://www.mathworks.it/products/simulink/.